Covers of Acts over Monoids

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1. **Preliminaries**
   - Acts
   - Classes of acts
2. **Known results on covers**
   - Coessential covers
   - Flat covers of modules
3. **$\mathcal{X}$-covers**
   - $\mathcal{X}$-precovers
   - $\mathcal{SF}/\mathcal{CP}$-covers
4. **Open problems**

Acts

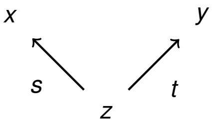
Given a monoid S, a (right) S-act is a set A equipped with an action

$$\begin{aligned} A \times S &\to A \\ (a, s) &\mapsto as, \end{aligned}$$

such that $(as)t = a(st)$ and $a1 = a$.

An S-act A is **cyclic** if it can be written in the form $A = aS$ for some $a \in A$.

An S-act A is **locally cyclic** if given any $x, y \in A$ there exits $z \in A$, $s, t \in S$ such that $x = zs$, $y = zt$ (equivalently, every finitely generated subact is contained in a cyclic subact).

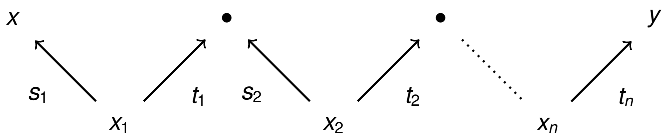


A monoid S satisfies **Condition** (A) if every locally cyclic S-act is cyclic, e.g. every finite monoid satisfies Condition (A).

Decomposition of acts

An S-act A is **decomposable** if it can be written as the coproduct of two subacts, $A = B \amalg C$, and **indecomposable** otherwise.

Given an indecomposable S-act A, for all $x, y \in A$, there exists $x_1, \ldots, x_n \in A$, $s_1, \ldots, s_n, t_1, \ldots, t_n \in S$ such that $x = x_1 s_1$, $x_1 t_1 = x_2 s_2, \ldots, x_n t_n = y$.

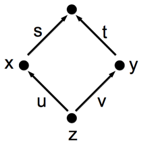


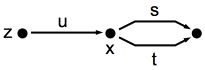
Theorem

Every S-act uniquely decomposes as a coproduct of indecomposable S-acts.

Flatness properties of acts

- An S-act X is **projective** if $Hom(X, -)$ preserves epimorphisms.
- An S-act X is **flat** if $X \otimes -$ preserves monomorphisms.
- An S-act X is **strongly flat** if $X \otimes -$ preserves pullbacks and equalizers, or equivalently if X satisfies **Conditions (P) and (E)**.
- An S-act satisfies **Condition (P)** if $xs = yt$ for $x, y \in X$, $s, t \in S$, then there exists $z \in X$, $u, v \in S$ such that $x = zu$, $y = zv$ and $us = vt$.
- An S-act satisfies **Condition (E)** if $xs = xt$ for $x \in X$, $s, t \in S$, then there exists $z \in X$, $u \in S$ such that $x = zu$ and $us = ut$.





Overview

Given a monoid S, let $\mathcal{P}$, $\mathcal{SF}$, $\mathcal{CP}$ and $\mathcal{F}$ be the classes of projective, strongly flat, Condition (P) and flat S-acts respectively.

Theorem

The following inclusions are valid and strict:

$$\mathcal{P} \subset \mathcal{SF} \subset \mathcal{CP} \subset \mathcal{F}.$$

Theorem

Let $\mathcal{X}$ be any of the following classes: $\mathcal{P}$, $\mathcal{SF}$, $\mathcal{CP}$ or $\mathcal{F}$, then $\coprod_{i\in I} X_i \in \mathcal{X}$ if and only if $X_i \in X$ for each $i \in I$.

Theorem

Let $\mathcal{X}$ be any of the following classes: $\mathcal{SF}$, $\mathcal{CP}$ or $\mathcal{F}$, then $\mathcal{X}$ is closed under direct limits.

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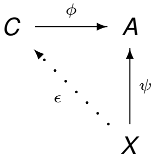
4 Open problems

Covers of acts

Let S be a monoid and $\mathcal{X}$ a class of S-acts closed under isomorphisms.

1. Given S-acts C and A, an epimorphism $\phi : C \to A$ is called coessential if given any proper subact $B \subseteq C$, $\phi|_B$ is not an epimorphism.
   If $C \in \mathcal{X}$, we call $\phi : C \to A$ an $\mathcal{X}$ **coessential cover** of A.
2. An homomorphism $\phi : C \to A$ with $C \in \mathcal{X}$ is called an $\mathcal{X}$-precover of A if every homomorphism $\psi : X \to A$ with $X \in \mathcal{X}$ can be factored through ϕ,



   and we call it an $\mathcal{X}$**-cover** of A whenever $\psi = \phi$ forces ϵ to be an isomorphism.

When $\mathcal{X} = \mathcal{P}$, the class of projective acts, then these are equivalent.

Results on covers

Modules over rings

projective $\Rightarrow$ flat

Acts over monoids

projective $\Rightarrow$ strongly flat $\Rightarrow$ condition (P) $\Rightarrow$ flat

Theorem, Bass 1960

Given any ring R, the following are equivalent:

- R is (right) perfect (every module has a projective cover).
- R satisfies DCC on principal (left) ideals.
- Every flat module is projective.

Results on covers

Modules over rings

projective $\Rightarrow$ flat

Acts over monoids

projective $\Rightarrow$ strongly flat $\Rightarrow$ condition (P) $\Rightarrow$ flat

Theorem, Fountain 1976

Given any monoid S, the following are equivalent:

- S is (right) perfect (every S-act has a projective cover).
- S satisfies DCC on principal (left) ideals and Condition (A).
- Every strongly flat act is projective.

Results on covers

Modules over rings

projective $\Rightarrow$ flat

Acts over monoids

projective $\Rightarrow$ strongly flat $\Rightarrow$ condition (P) $\Rightarrow$ flat

Question

When do modules have flat coessential covers?

'On generalized perfect rings' (Amini, Amini, Ershad, Sharif, 2007)

Results on covers

Modules over rings

projective $\Rightarrow$ flat

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Question

When do modules have flat coessential covers?

'On generalized perfect rings' (Amini, Amini, Ershad, Sharif, 2007)
$\mathbb{Z}/n\mathbb{Z}$ does not have a flat coessential cover as a $\mathbb{Z}$-module.

Results on covers

Modules over rings

projective $\Rightarrow$ flat

Acts over monoids

projective $\Rightarrow$ strongly flat $\Rightarrow$ condition (P) $\Rightarrow$ flat

Theorem (Khosravi, Ershad, Sedaghatjoo, 2009)

Given a monoid S the following are equivalent:

- Every S-act has a strongly flat coessential cover.
- Every left unitary submonoid T of S contains a left collapsible submonoid R such that for each $u \in T$, $uS \cap R \neq \emptyset$, and S satisfies Condition (A).

Results on covers

Modules over rings

projective $\Longrightarrow$ flat

Acts over monoids

projective $\Longrightarrow$ strongly flat $\Longrightarrow$ condition (P) $\Longrightarrow$ flat

Theorem (Khosravi, Ershad, Sedaghatjoo, 2009)

Given a monoid S the following are equivalent:

- Every S-act has a Condition (P) coessential cover.
- Every left unitary submonoid T of S contains a right reversible submonoid R such that for each $u \in T$, $uS \cap R \neq \emptyset$, and S satisfies Condition (A).

Results on covers

Modules over rings

projective $\Rightarrow$ flat

Acts over monoids

projective $\Rightarrow$ strongly flat $\Rightarrow$ condition (P) $\Rightarrow$ flat

Question

When do acts have flat coessential covers?

Flat covers of acts

In 1981 Enochs introduced the idea of an $\mathcal{X}$-cover.

- He showed that if a module has an $\mathcal{F}$-precover then it has an $\mathcal{F}$-cover. (Any class closed under direct limts.)
- He also conjectured that every module has an $\mathcal{F}$-cover. This came to be known as the flat cover conjecture.
- In 1995 J. Xu proved that $\mathcal{F}$-covers always exist for certain types of commutative Noetherian rings.
- The conjecture was finally proved independently by Enochs and Bican & El Bashir and published in a joint paper in 2001.

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$\mathcal{X}$-precovers

Theorem (B & R, 2011)

Let S be a monoid, and $\mathcal{X}$ a class of S-acts closed under direct limits. If an S-act A has an $\mathcal{X}$-precover, then A has an $\mathcal{X}$-cover.

Since strongly flat, Condition (P), and flat acts are all closed under direct limits...

Corollary

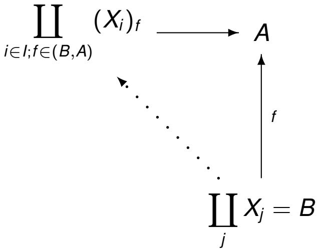
If A has an $\mathcal{SF}/\mathcal{CP}/\mathcal{F}$-precover then it has an $\mathcal{SF}/\mathcal{CP}/\mathcal{F}$-cover.

Theorem (Bridge, 2010 (Teply, 1976))

Let $\mathcal{X}$ be a class of S-acts such that $\coprod_{i\in I} X_i \in \mathcal{X} \Leftrightarrow X_i \in \mathcal{X}$ for each $i \in I$. Let S be a monoid that has only a set of indecomposable S-acts with property $\mathcal{X}$, then every S-act has an $\mathcal{X}$-precover.

Sketch proof

Let $\{X_i : i \in I\}$ be a set of indecomposable S-acts with property $\mathcal{X}$, and let each $(X_i)_f \cong X_i$. Then we have the following $\mathcal{X}$-precover.



Results

Theorem (B & R, 2011)

Let S be a monoid that satisfies Condition (A), then every S-act has an $\mathcal{SF}/\mathcal{CP}$-cover.

Proof.

- Every Condition (P) act is a coproduct of locally cyclic acts
- Condition (A) $\Leftrightarrow$ locally cyclic acts are cyclic
- The cardinality of a cyclic act is bounded $|S/\rho| \leq |S|$
- The class of all indecomposable Condition (P) acts is a set

□

Corollary

If every S-act has an $\mathcal{SF}/\mathcal{CP}$ coessential cover then it has an $\mathcal{SF}/\mathcal{CP}$-cover.

Results

Example (B & R, 2011)

There exist monoids that have a proper class (not a set) of indecomposable strongly flat acts, for example the full transformation monoid of an infinite set.

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Open problems

Question

Does every S-act have an $\mathcal{SF}$-cover?

Question

If an S-act has an $\mathcal{X}$ coessential cover, does it have an $\mathcal{X}$-cover?

Question

What about other classes of acts, e.g. injective? (Enochs showed that every module has an injective cover if and only if the ring is Noetherian.)

Question

What about a dual theory for envelopes?